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# Mean-field approximations for the restricted solid-on-solid growth models

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Received 2 March 2007, in final form 25 May 2007

Published 3 July 2007

Online at [stacks.iop.org/JPhysA/40/8205](http://stacks.iop.org/JPhysA/40/8205)

## Abstract

We study models for surface growth with a wetting and a roughening transition using simple and pair mean-field approximations. The simple mean-field equations are solved exactly and they predict the roughening transition and the correct growth exponents in a region of the phase diagram. The pair mean-field equations, which are solved numerically, show a better accordance with numerical simulation and correctly predict a growing interface with a constant velocity at the moving phase. Also, when detailed balance is fulfilled, the pair mean field becomes the exact solution of the model.

PACS numbers: 05.70.Ln, 05.70.Np, 68.08.Bc

## 1. Introduction

Surface growing [1, 2] is a phenomenon observed in nature as well as in the laboratory. As an example of the latter we cite the experimental technique known as the molecular beam epitaxy which allows the growing of the surface at the atomic level. The surface is described by a height variable  $h(\mathbf{r}, t)$  that gives the height of the deposited layer at a given point  $\mathbf{r}$  of the substrate at time  $t$ . Several models have been introduced to describe the mean features of surface growing in which the heights are stochastic variables whose time evolution is governed by a Markovian stochastic process. Here, we are concerned with the solution of such models by the use of mean-field approximations.

Two relevant quantities are used to characterize the surface growth. One is the mean height  $\bar{h}$  of the surface from the substrate and the other is the surface width  $w$ , which is a measure of the surface roughness. The divergence of  $w$  characterizes a rough interface. According to Family and Vicsek [3], the width of a rough surface of a sufficient large system behaves as

$$w \sim t^\gamma, \quad (1)$$

where  $\gamma$  is the growth exponent. This behaviour characterizes a rough thermodynamic phase. Otherwise, that is, if the width of an infinite system remains finite when  $t \rightarrow \infty$ , the surface is smooth. A roughening transition takes place when, by varying the control parameters, the surface changes from a smooth to a rough surface.

The interface may yet be pinned or moving. A moving thermodynamic phase is characterized by a constant velocity  $v$  of the interface, or in other words, by a linear growth of the mean height, that is

$$\bar{h} = vt. \quad (2)$$

A depinning transition [4] from a pinned to a moving phase occurs when, by changing the control parameters, the interface begins to move with a constant velocity. This is also known as a wetting transition [5] since in the moving phase the mean height becomes infinitely large when  $t \rightarrow \infty$ , which characterizes a wet phase. At the depinning transition, the mean height may not grow linearly with time but may behave according to

$$\bar{h} \sim t^\gamma. \quad (3)$$

In the model we study here the depinning transition coincides with the roughening transition.

If we define  $P(h, t)$  as the one-point height probability distribution at time  $t$ , the mean height  $\bar{h}$  and the square of the surface width  $w^2$  are the first moment and the variance of this probability distribution, respectively. The asymptotic behaviour given by equations (1) and (2) can then be obtained from the following scaling form [6]:

$$P(h, t) = t^{-\gamma} f\left(\frac{h - vt}{t^\gamma}\right), \quad (4)$$

where  $f(x)$  is a universal function. The behaviour given by equations (1) and (3) also follows from this same form by setting  $v = 0$ .

Some models have been introduced to describe the surface growth and the asymptotic behaviour in which the heights are the stochastic variables governed by Langevin equations. Two important ones are the Edwards–Wilkinson (EW) [7]

$$\frac{\partial h}{\partial t} = v\nabla^2 h + \eta(\mathbf{r}, t) \quad (5)$$

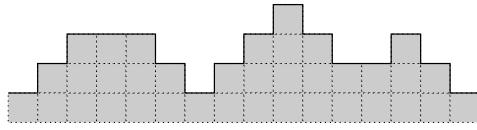
and the Kardar–Parisi–Zhang (KPZ) [8]

$$\frac{\partial h}{\partial t} = v\nabla^2 h + \lambda(\nabla h)^2 + \eta(\mathbf{r}, t), \quad (6)$$

where  $\eta(\mathbf{r}, t)$  is a white noise. Since the EW equation is linear, it stays invariant under the transformation  $h \rightarrow -h$  while the KPZ equation, which is nonlinear, lacks this property. In one dimension the growth exponent for the EW class is  $\gamma = 1/4$  and for the KPZ class is  $\gamma = 1/3$ .

In this work we study a growth model, introduced by Hinrichsen *et al* [9], with deposition and evaporation of particles that respects the restricted solid-on-solid (RSOS) condition [10] and in which the evaporation at the initial height is forbidden, that is, there is a wall at the zero height. The presence of the wall leads to a nonequilibrium wetting transition depending on the evaporation and deposition rates. It is worth mentioning that nonequilibrium wetting transition has been already studied by another approach, namely by mapping the KPZ with a potential into a Langevin equation with a multiplicative noise [11, 12].

The growth model is studied here by means of simple and pair mean-field approximations. The mean-field approximation at the pair level, which does respect the RSOS condition exactly, is found to be capable of describing a moving phase, that is, a moving interface at constant



**Figure 1.** A configuration of heights with the RSOS condition.

velocity. Another feature of the two-site mean-field approximation is that it becomes the exact solution when detailed balance is fulfilled.

Recently, a mean-field theory for surface growth has been introduced by Hinrichsen *et al* [13] and used by Ginelli and Hinrichsen [6] to study a single-step model for surface growth. The one-site and two-site mean-field approximations we use here are distinct from that of Hinrichsen *et al* [13], but share properties that are similar.

The paper is organized as follows. In section 2, we define the model to be examined and write down the master equation for general models with the RSOS condition. In section 3, we solve the master equation within the one-site mean-field approach exactly. Section 4 is dedicated to the pair mean-field approach. The resultant equations are solved numerically and compared with the results obtained from the one-site approximation and from simulations. In section 5 we analyse the detailed balance condition and in section 6 we present our conclusions.

## 2. Model and master equation

We consider a one-dimensional lattice where a discrete variable  $h_i$  is attached to the site  $i$ , which represents the height of the pile of particles at site  $i$ . The stochastic variable  $h_i$  takes the integer values. At time  $t = 0$  we consider a flat surface at the zero level, that is,  $h_i = 0$  for all  $i$ . We consider models with random sequential updates of the heights, which have not only deposition but also evaporation. We will restrict ourselves only to models that obey the RSOS condition, that is, such that  $|h_i - h_{i+1}| \leq 1$  (see figure 1). At each time step only one site is updated according to local stochastic rules. If site  $i$  is chosen, these rules will affect only the neighbouring sites  $i + 1$  and  $i - 1$  and site  $i$  itself. The rate of the transition  $h_i \rightarrow h_i + n$  is denoted by  $c_n(h_{i-1}, h_i, h_{i+1})$ . The only possible transitions are those for which  $n = \pm 1$  meaning that the height is increased (deposition) or decreased (evaporation) by just one unit.

The possible transition rates are presented in figure 2 and are given by

$$c_+(h, h, h) = p_1, \quad (7)$$

$$c_+(h, h, h + 1) = c_+(h + 1, h, h) = p_3, \quad (8)$$

$$c_+(h + 1, h, h + 1) = p_5, \quad (9)$$

$$c_-(h - 1, h, h - 1) = p_2, \quad (10)$$

$$c_-(h - 1, h, h) = c_-(h, h, h - 1) = p_4, \quad (11)$$

$$c_-(h, h, h) = p_6, \quad (12)$$

and the model so defined has six parameters:  $p_1, \dots, p_6$ . By rescaling time we see that they are not all independent and one of them can set equal to unity.

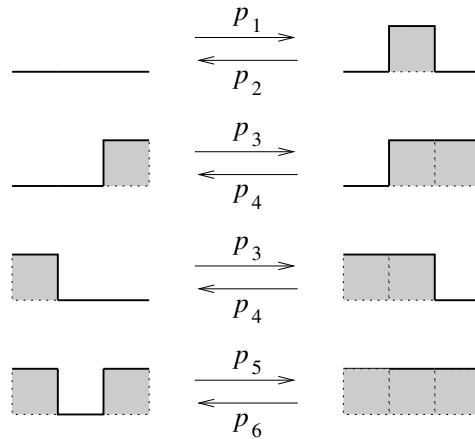


Figure 2. Transition rates.

The probability  $P(h_1, h_2, \dots, t)$  of a given configuration  $(h_1, h_2, \dots)$  at time  $t$  obeys the master equation

$$\frac{d}{dt} P(h_1, h_2, \dots, t) = \sum_{n=\pm 1} \sum_i \{c_n(h_{i-1}, h_i - n, h_{i+1}) P(h_1, \dots, h_i - n, \dots, t) - c_n(h_{i-1}, h_i, h_{i+1}) P(h_1, \dots, h_i, \dots, t)\}, \quad (13)$$

and we are considering periodic boundary conditions. For later use we write down the time evolution of the marginal probability distribution related to a given site, say site 1. It is given by

$$\frac{d}{dt} P(h_1, t) = \sum_{h_0, h_2} \sum_{n=\pm 1} \{c_n(h_0, h_1 - n, h_2) P(h_0, h_1 - n, h_2, t) - c_n(h_0, h_1, h_2) P(h_0, h_1, h_2, t)\}. \quad (14)$$

We also write down the time evolution of the marginal probability distribution related to two consecutive sites, say sites 1 and 2. It is given by

$$\frac{d}{dt} P(h_1, h_2, t) = 2 \sum_{h_0} \sum_{n=\pm 1} \{c_n(h_0, h_1 - n, h_2) P(h_0, h_1 - n, h_2, t) - c_n(h_0, h_1, h_2) P(h_0, h_1, h_2, t)\}, \quad (15)$$

valid for solutions that are translationally invariant, where we have made use of the symmetry  $c_n(h_0, h_1, h_2) = c_n(h_2, h_1, h_0)$ .

For certain values of the parameters the stochastic process exhibits detailed balance. This means to say that each term of the summation on the right-hand side of equation (13) vanishes in the time-independent stationary state. In this case, the time-independent stationary probability distribution can be written as the product

$$P(h_1, h_2, h_3, h_4, \dots) = \frac{1}{Z} T(h_1, h_2) T(h_2, h_3) T(h_3, h_4) \dots \quad (16)$$

where  $T(h_1, h_2)$  are the elements of a symmetric matrix  $T$ , which vanish whenever the RSOS condition is not fulfilled. The detailed balance condition gives the following relations for the nonvanishing elements of  $T$ , denoted by  $X_h = T(h, h)$  and  $Y_h = T(h, h + 1)$ ,

$$p_1 X_h^2 = p_2 Y_h^2, \tag{17}$$

$$p_3 X_h Y_h = p_4 Y_h X_{h+1}, \tag{18}$$

$$p_5 Y_h^2 = p_6 X_{h+1}^2. \tag{19}$$

From these equations follows the relation [14]

$$\frac{p_1 p_5}{p_2 p_6} = \frac{p_3^2}{p_4^2}, \tag{20}$$

which is the condition for detailed balance to hold.

In the specific model [9] we study here, the deposition occurs at a rate  $q$  and the evaporation, for nonzero heights, occurs at rate  $r$  or  $p$ , depending on the neighbours configuration. More specifically, if  $h_i = 0$ ,  $p_1 = p_3 = p_5 = q$  and  $p_2 = p_4 = p_6 = 0$ ; and if  $h_i \neq 0$ ,

$$p_1 = p_3 = p_5 = q, \quad p_2 = p_4 = r, \quad p_6 = p. \tag{21}$$

The detailed balance condition (20) gives  $p = r$ . Therefore, when  $p = r$  is respected the model can be solved exactly [9] for the pinned phase.

Without the wall at the initial height, we would have a rough interface growing in the positive direction for  $q > q_c$  and in the negative direction for  $q < q_c$ . With the wall the phase  $q > q_c$  is not affected, in the sense that the interface still grows and is rough. For  $q < q_c$ , in the presence of the wall, the interface is smooth and stays pinned to the substrate. Hence, at  $q = q_c$ , the model displays a depinning and a roughening transition. The rough interface may display a EW or a KPZ behaviour. In the case  $p = r$ , it is known that the crossover from EW to KPZ behaviour [15] coincides with the critical point occurring at  $q = p$ .

The case  $p = 0$  [16, 17] is special since in this case no particle can be evaporated from a completed filled layer and the presence of the wall thus makes no difference. The critical behaviour at  $p = 0$  places the model in the universality class of the directed percolation (DP) class [18, 19].

The order parameter of the pinned phase is the density of sites,  $P_0$ , in contact with the substrate or the wall. It is assumed to behave near and below the transition as

$$P_0 \sim (q_c - q)^\beta. \tag{22}$$

The interface width  $w$  is finite at the pinned phase but diverges as one approaches the critical point. We assume that it diverges according to

$$w \sim (q_c - q)^{-\zeta}. \tag{23}$$

The order parameter of the moving phase is chosen to be the velocity  $v$  of the interface growing, defined by

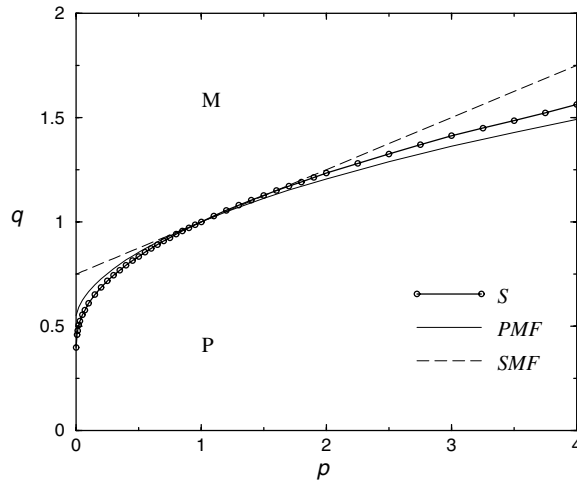
$$v = \frac{d}{dt} \langle h_1 \rangle = \sum_{h_1} h_1 \frac{d}{dt} P(h_1, t), \tag{24}$$

which can also be written as

$$v = \sum_{h_0, h_1, h_2} \sum_{n=\pm 1} n c_n(h_0, h_1, h_2) P(h_0, h_1, h_2, t). \tag{25}$$

Near and above the transition we assume that the velocity behaves as

$$v \sim (q - q_c)^\theta. \tag{26}$$



**Figure 3.** Phase diagram for  $r = 1$  in the  $p$ - $q$  plane. The critical lines, separating the moving (M) and pinned (P) phases, were determined by simple mean-field (SMF), pair mean-field (PMF) approximation and numerical simulations (S).

### 3. Simple mean-field approximation

We are interested in studying the solution of the master equation associated with an infinite system. In order to solve the master equation (13) we begin by using a simple mean-field approximation. In this approach all the correlations are neglected resulting in the following approximation:  $P(h_{i-1}, h_i, h_{i+1}, t) = P(h_{i-1}, t)P(h_i, t)P(h_{i+1}, t)$ . Insertion of this approximation into equation (14) yields a closed equation for the one-site probability  $P_k(t)$  distribution

$$\begin{aligned} \frac{d}{dt} P_k = & q(P_{k-1}^3 - P_k^3) + (r - 2q)(P_k^2 P_{k+1} - P_{k-1}^2 P_k) \\ & + (2r - q)(P_k P_{k+1}^2 - P_{k-1} P_k^2) + p(P_{k+1}^3 - \gamma_k P_k^3), \end{aligned} \quad (27)$$

valid for the specific model defined by the rates (21), where we are using the integer variable  $k$  in the place of the height  $h$  and initially  $P_0 = 1$  and  $P_k = 0$  for  $k \neq 0$  because we are considering a initial flat surface. The variable  $\gamma_k = 0$  when  $k = 0$  and  $\gamma_k = 1$  otherwise. The equation is valid for  $k > 0$  and for  $k = 0$  provided we set  $P_{-1} = 0$  on the right-hand side.

Let us first look for a possible stationary state, that is, a time-independent solution, which corresponds to a pinned phase. If we assume a solution of the form

$$P_k = A\lambda^k, \quad (28)$$

we can check easily by substitution that this is indeed a solution provided  $\lambda$  is the root of the third-order algebraic equation

$$-p\lambda^3 + (q - 2r)\lambda^2 + (2q - r)\lambda + q = 0. \quad (29)$$

The critical line, shown in figure 3, is found by letting  $\lambda \rightarrow 1$  with the result

$$q_c = \frac{1}{4}(p + 3r), \quad (30)$$

and the stationary solution occurs for  $q < q_c$ . The normalization of  $P_k$ , given by (28), gives  $A = 1 - \lambda$  so that  $P_0 = 1 - \lambda$ . The mean height  $\bar{h}$  and the square of the interface width  $w^2$  are determined as the average and the variance of the distribution given by (28), that is

$$\bar{h} = \langle k \rangle \quad \text{and} \quad w^2 = \langle k^2 \rangle - \bar{h}^2, \tag{31}$$

resulting in  $\bar{h} = \lambda/(1 - \lambda)$  and  $w = \sqrt{\lambda}/(1 - \lambda)$ . Near the critical line,  $\lambda$  approaches 1 as  $1 - \lambda = 2(q_c - q)/(r + p)$  from which follow the results  $P_0 \sim (q_c - q)$ ,  $\bar{h} \sim (q_c - q)^{-1}$  and  $w \sim (q_c - q)^{-1}$ .

We solve the equation for  $q \geq q_c$  by a method similar to that of Ginelli and Hinrichsen [6] in which a continuous version of the equation is used. Writing  $h = k$ , the equation for the probability distribution  $P(h, t)$  becomes, up to second order,

$$\frac{\partial P}{\partial t} = -aP^2 \frac{\partial P}{\partial h} + b \left[ P \left( \frac{\partial P}{\partial h} \right)^2 + \frac{1}{2} P^2 \frac{\partial^2 P}{\partial h^2} \right], \tag{32}$$

where

$$a = 12(q - q_c) \tag{33}$$

and

$$b = 2q + r + 3p. \tag{34}$$

At the critical line,  $q = q_c$ , the coefficient  $a$  vanishes and we end up with the equation

$$\frac{\partial P}{\partial t} = b \left[ P \left( \frac{\partial P}{\partial h} \right)^2 + \frac{1}{2} P^2 \frac{\partial^2 P}{\partial h^2} \right], \tag{35}$$

which can be solved by assuming the scaling form (4) for  $P(h, t)$ . A consistency is achieved only if we choose the growth exponent as being  $\gamma = 1/4$ . Along the critical line, the simple mean-field approximation is then compatible with an EW behaviour. The substitution of the scaling form into the equation leads to the following equation for the scaling function  $f(x)$ :

$$f(x) + xf'(x) + 4bf(x)[f'(x)]^2 + 2b[f(x)]^2 f''(x) = 0, \tag{36}$$

whose solution is

$$f(x) = \sqrt{1 - \frac{x^2}{2b}}. \tag{37}$$

The asymptotic probability distribution, valid for large times, can then be written as

$$P(h, t) = \sqrt{\frac{1}{t^{1/2}} - \frac{h^2}{2bt}}. \tag{38}$$

For  $q > q_c$  it suffices to consider only the linear term in  $b$ . The equation for  $P(h, t)$  reads

$$\frac{\partial P}{\partial t} = -aP^2 \frac{\partial P}{\partial h}, \tag{39}$$

where now  $a$  is strictly positive. This equation can be solved by assuming again a scaling relation of the form (4). Now, however, the consistency requires an exponent  $\gamma = 1/3$  so that the simple mean-field approximation is then compatible with a KPZ behaviour. The substitution gives the following equation for the scaling function:

$$f(x) + xf'(x) = 3a[f(x)]^2 f'(x), \tag{40}$$

whose solution is

$$f(x) = \sqrt{\frac{x}{a}}. \tag{41}$$



The asymptotic probability distribution, valid for large time, can then be written as

$$P(h, t) = \sqrt{\frac{h}{at}}. \quad (42)$$

We remark that  $P(h, t)$  vanishes for values of  $h$  larger than a certain  $h_{\max}$  which depends on time. This maximum value of  $h$  is determined by the normalization of  $P(h, t)$  and is given by  $h_{\max} = (9at/4)^{1/3}$ . This allows us to determine  $\bar{h} = 3h_{\max}/5$  and  $w = (12/175)^{1/2}h_{\max}$  so that

$$\bar{h} \sim w \sim (q - q_c)^{1/3}t^{1/3}. \quad (43)$$

Within the simple mean-field approximation we were able to observe a roughening transition occurring at the transition line shown in figure 3. Nevertheless, this approximation is not able to predict a moving surface with a constant velocity. This deficiency will be overcome in the next section by means of the pair mean-field approximation.

#### 4. Pair mean-field approximation

In the pair mean-field approach we solve equation (15) by using the approximation for the three-site probability distribution

$$P(h_1, h_2, h_3, t) = \frac{P(h_1, h_2, t)P(h_2, h_3, t)}{P(h_2, t)}, \quad (44)$$

where  $P(h_1, h_2, t)$  and  $P(h_2, t)$  are the two-site and one-site probability distribution, respectively. These two quantities are related by

$$P(h_2, t) = \sum_{h_1} P(h_1, h_2, t) = \sum_{h_3} P(h_2, h_3, t). \quad (45)$$

Due to the RSOS condition there are actually three types of two-site probabilities:  $P_{k,k}(t)$ ,  $P_{k,k-1}(t)$  and  $P_{k,k+1}(t)$ , where as before we are using the integer variable  $k$  in the place of the height  $h$ . Together with the one-site probability  $P_k$  they form a set of four variables. However they are not all independent since  $P_{k+1,k} = P_{k,k+1}$  and

$$P_k = P_{k,k} + P_{k,k-1} + P_{k,k+1}. \quad (46)$$

Therefore we are left with two independent variables, for each  $k$ , which we choose to be  $P_{k,k}$  and  $P_{k,k+1}$ . Initially we have  $P_{k,k+1} = 0$  for all  $k$  and  $P_{k,k} = 0$  for  $k \neq 0$  and  $P_{0,0} = 1$ .

To simplify notation, we denote  $P_{k,k+1}$  by  $y_k$  and  $P_{k,k}$  by  $x_k$  so that

$$P_k = x_k + y_k + y_{k-1}. \quad (47)$$

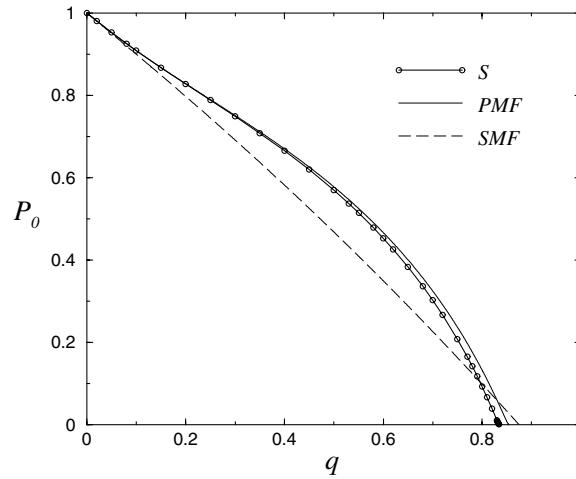
The pair mean-field equations for  $x_k$  and  $y_k$ , corresponding to the model defined by (21), then read

$$\frac{d}{dt}x_k = 2q \left[ \frac{y_{k-1}(y_{k-1} + x_{k-1})}{P_{k-1}} - \frac{x_k(x_k + y_k)}{P_k} \right] + 2r \left[ \frac{y_k(y_k + x_{k+1})}{P_{k+1}} - \frac{x_k y_{k-1}}{P_k} \right] - 2p \frac{x_k^2}{P_k} \gamma_k, \quad (48)$$

where we should set  $x_{-1} = y_{-1} = 0$  when  $k = 0$  and

$$\frac{d}{dt}y_k = q \frac{x_k^2 - y_k^2}{P_k} - r \frac{y_k^2}{P_{k+1}} + p \frac{x_{k+1}^2}{P_{k+1}}. \quad (49)$$

The numerical integration of the coupled equations (48) and (49), performed by repeated iteration from the initial condition  $x_k = 0$  for  $k \neq 0$ ,  $x_0 = 1$  and  $y_k = 0$  for all  $k$ , allows us to determine the one-site probability distribution (47). From the obtained distribution we get the



**Figure 4.** Order parameter  $P_0$  as a function of  $q$  for  $p = 0.5$  obtained from simple mean-field (SMF), pair mean-field (PMF) approximation and numerical simulations (S) performed with 2048 sites and 100 independent realizations. The critical values are  $q_c = 0.875$  for SMF,  $q_c = 0.8539$  for PMF and  $q_c = 0.8346(1)$  for the simulations.

mean height and the interface width by the use of equations (31) and the interface velocity by derivating numerically the mean height with respect to time.

The transition line is shown in figure 3. For comparison we also show the transition line obtained from numerical simulation. Below the transition line, in the pinned phase, the density of sites  $P_0$  in contact with the substrate is found to behave as

$$P_0 \sim (q_c - q), \quad (50)$$

so that the exponent  $\beta = 1$ , the same value found in the one-site mean-field approximation. The width of the interface diverges at the critical line as

$$w \sim (q_c - q)^{-1/3}, \quad (51)$$

yielding an exponent  $\zeta = 1/3$ .

As an example, we show in figures 4 and 5, respectively, the order parameter  $P_0$  and the width  $w$  as functions of the deposition rate  $q$ , at  $p = 0.5$ . We see that  $P_0$  is linear near the transition point and that  $w^{-3}$  is linear supporting behaviour (51).

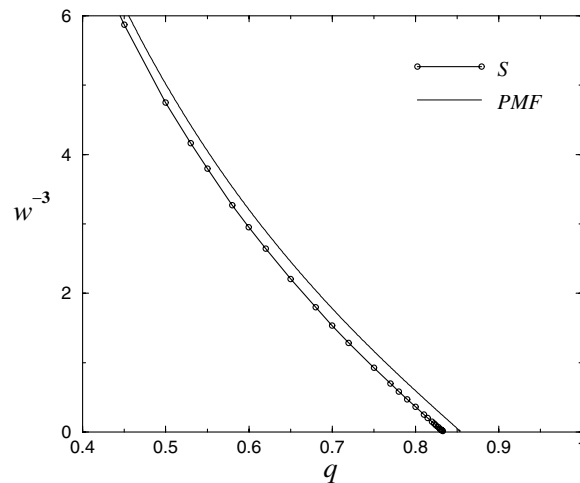
Above and at the transition line, the numerical integration gives a time-dependent probability distribution  $P_k(t)$  from which we calculate the average height and the width of the interface. Along the transition line we found that

$$w \sim t^{1/4}, \quad q = q_c, \quad (52)$$

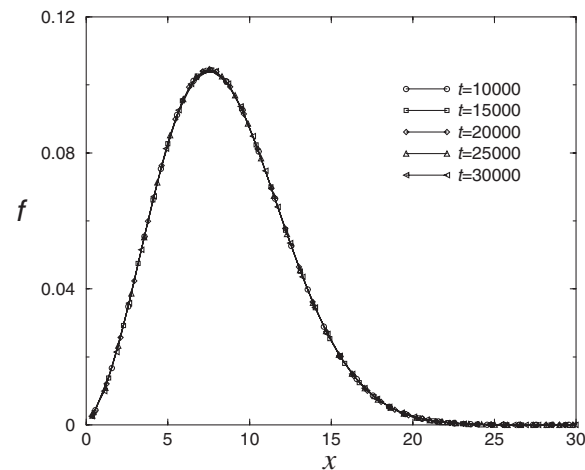
and above it

$$w \sim t^{1/3}, \quad q > q_c. \quad (53)$$

In figures 6 and 7 we show a data collapse of  $P_k(t)$  by using the scaling form (4) at the critical point ( $p = 1$  and  $q = 1$ ) and above the critical point, inside the moving phase ( $p = 1$  and  $q = 2$ ). In the former case we have used  $\gamma = 1/4$  and in the latter,  $\gamma = 1/3$ . The good data collapse leads us to the results (52) and (53).



**Figure 5.** Interface width  $w$  as a function of  $q$  for  $p = 0.5$  obtained from pair mean-field (PMF) approximation and numerical simulations (S) performed with 2048 sites and 100 independent realization. The critical values are  $q_c = 0.8539$  and  $q_c = 0.8346(1)$ , respectively.

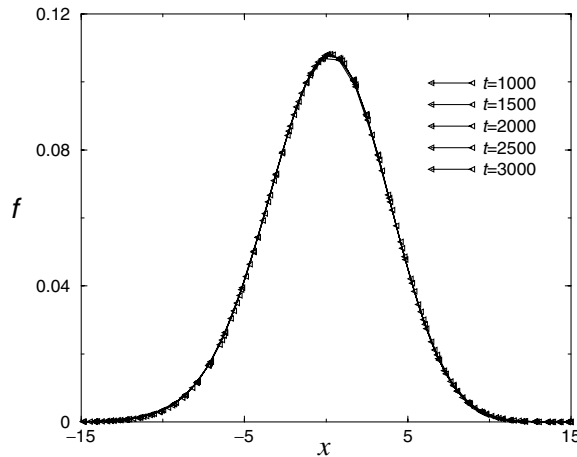


**Figure 6.** Scaling function  $f = P(h, t)t^{1/4}$  versus  $x = ht^{-1/4}$  as obtained from the pair mean-field approximation for the values of  $t$  shown in the figure, at the critical point  $q = p = 1$ . The function  $f(x)$  extrapolates to zero when  $x \rightarrow 0$ .

The asymptotic results for the width of the interface are the same as those found by means of the one-site mean-field approximation. However, as already mentioned, the two-site approximation is capable of describing a moving interface. We found that, inside the moving phase the mean height increases with a constant velocity  $v$  so that the pair mean-field approximations predict correctly a linear growth of the moving interface. As we approach the critical line the velocity vanishes according to

$$v \sim (q - q_c), \quad (54)$$

that is,  $\theta = 1$ .



**Figure 7.** Scaling function  $f = P(h, t)t^{1/3}$  versus  $x = (h - vt)t^{-1/3}$  as obtained from the pair mean-field approximation for the values of  $t$  shown in the figure, inside the moving phase at the point  $q = 2$  and  $p = 1$ . The velocity is found to be  $v = 0.112$ .

As we have seen, the pair mean-field approximation takes into account the correlation of two consecutive sites which results in the exact satisfaction of the RSOS condition in opposition to the simple mean-field approximation. Because of that the pair mean-field results are closer to the simulation ones when compared to the simple mean-field results. For instance, the transition line is in general closer to the simulation results and has the right concavity.

### 5. Detailed balance

When detailed balance is obeyed, it turns out that the two-site approximation becomes an exact solution. In this case, the stationary probability distribution is of the form (16) and the elements of  $X_k$  and  $Y_k$  of the transfer matrix  $T$  are related to  $x_k$  and  $y_k$  by  $x_k = X_k P_k / \Lambda$  and  $y_k = Y_k \sqrt{P_k P_{k+1}} / \Lambda$ , where  $\Lambda$  is the dominant eigenvalue of  $T$ . The detailed balance conditions (17)–(19) are then equivalent to the following relations:

$$p = r = 1, \quad q \frac{x_k}{P_k} = \frac{x_{k+1}}{P_{k+1}}, \quad y_k^2 = x_k x_{k+1}, \tag{55}$$

from which follows the solution

$$\frac{x_k}{P_k} = \frac{1}{\lambda} q^k, \quad \frac{y_k}{\sqrt{P_k P_{k+1}}} = \frac{1}{\lambda} q^{k+1/2}, \tag{56}$$

where  $\lambda$  is a constant to be found. That this is indeed a solution of the pair mean-field equations can be checked by substitution.

Inserting these relations into equation (47) we get the following eigenvalue equation for  $\phi_k = \sqrt{P_k}$ :

$$q^{k-1/2} \phi_{k-1} + q^k \phi_k + q^{k+1/2} \phi_{k+1} = \lambda \phi_k \tag{57}$$

so that the constant  $\lambda$  is identified as the eigenvalue, actually, the dominant eigenvalue  $\Lambda$ . The dominant eigenfunction gives the one-site probability distribution  $P_k = \phi_k^2$ . The eigenvalue equation (57) was found by Hinrichsen *et al* [13] who solved it in the vicinity of the critical

point by using a continuous height approximation. Their solution gives the following results in the vicinity of the critical point  $q = q_c = 1$ :

$$w \sim (q_c - q)^{-1/3} \quad (58)$$

and

$$P(0) \sim (q_c - q). \quad (59)$$

These results are the same results found in the previous section by numerical integration of the two-site pair approximation equations.

## 6. Conclusion

We have studied a lattice model for surface growth by means of one-site and two-site mean-field approximations. Within the simple mean-field approximation, whose equations were solved exactly, we were able to observe a roughening transition occurring with growth exponents equal to  $1/4$  at the transition line and  $1/3$  inside the rough phase. This simple approximation was not able to predict a moving surface with a constant velocity. However, this deficiency was overcome by means of the pair mean-field approximation which predicts correctly a linear growth of the moving interface. In general, the pair approximation gives results that are in better accordance with numerical simulations. The critical line is closer to that given by numerical simulations. The growth exponent is  $1/4$  at the critical line and  $1/3$  inside the moving phase. At  $p = 1$  these approximations correctly predict the crossover from the EW, at  $q = 1$ , to KPZ, for  $q > 1$ .

The pair mean-field approximations gives also the following results. As one approaches the critical line from inside the pinned phase, the order parameter vanishes with an exponent  $\beta = 1$  and the width of the interface diverges with an exponent  $\zeta = 1/3$ . If one approaches the line from the moving phase, the velocity of the interface vanishes with an exponent  $\theta = 1$ .

The pair mean-field approximation has two important features: it takes into account the RSOS condition exactly and reduces to the exact solution when detailed balance is fulfilled. This last feature indicates that the pair mean-field is a good approximation for the nonequilibrium case ( $p \neq 1$ ), where exact results are generally not known. This approximation can also be used to get the properties and the phase diagram of the general model defined in figure 2, with or without a wall. For growth models that do not respect the RSOS condition, in principle, the pair approximation can also be used but the equations will have a more cumbersome form.

## Acknowledgments

This research was supported by the Brazilian agency CNPq.

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